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Damped Oscillatory Integral Operators with Analytic Phases*

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I. INTRODUCTION

The study of oscillatory integral operators with degenerate phase functions has attracted considerable interest lately. While the general higher dimensional problem seems out of reach at the present, much progress has been made in the one-dimensional situation. In this case, it is now known that the decay rate for the usual oscillatory integral operator is determined exactly by the reduced Newton diagram of the phase (in the analytic case). An intriguing issue that remained is the corresponding decay rate for the "damped" oscillatory integral operator and in particular the value of the critical damping exponent. The purpose of this paper is to solve this problem.

We begin by recalling the results of [5], the main goal of which was the study of oscillatory integral operators with analytic phases of arbitrarily high degrees of degeneracy. Let T_λ be an operator of the form

$$T_\lambda f(x) = \int_{-\infty}^{\infty} e^{i\lambda S(x,\,y)} \chi(x,\,y)\, f(y)\, \mathrm{d}y, \qquad f \in C_0^\infty(\mathbf{R}), \tag{1.1}$$

where $S(x, y)$ is real and analytic, and the support of the amplitude $\chi(x, y)$ is a sufficiently small neighborhood of the origin. In [5], we had established the sharp estimate $\|T_\lambda\| \leqslant C\,|\lambda|^{-\delta/2}$, where the decay rate δ is given by the

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so-called *Newton decay rate* of $S(x, y)$ at the origin. Recall that the Newton decay rate δ is characterized by the fact that $(\delta^{-1}, \delta^{-1})$ is the intersection of the bisectrix $p=q$ with the boundary of the reduced Newton diagram of the phase $S(x, y)$ at the origin, and that the reduced Newton diagram of $S(x, y)$ is constructed in the same way as the usual Newton diagram, after removing the vertices on both p and q axes. In particular $\delta=1$ only when the Hessian S''_{xy} is non-zero. In the present paper, we shall show that, in contrast to T_λ, the suitably damped operator D defined by

$$Df(x)=\int_{-\infty}^{\infty} e^{i\lambda S(x,\,y)}\,|S''_{xy}(x, y)|^{1/2}\,\chi(x, y)\,f(y)\,\mathrm{d}y, \qquad f\in C_0^\infty(\mathbf{R}) \tag{1.2}$$

always achieves the optimal decay rate $\delta=1$. More precisely, we have:

THEOREM 1. *Assume that the phase function $S(x, y)$ is a real and analytic function, and that $\chi(x, y)$ is a C^2 function with compact support. Then*

$$\|D\| \leqslant C\,|\lambda|^{-1/2}. \tag{1.3}$$

Remarkably, damped operators of the form (1.2) are appearing with increasing frequency in a number of contexts, including L^p-L^q regularity for Radon transforms [4], and global existence theorems for non-linear wave equations [2]. Some related and earlier work in this direction is in [1] and [8].

It is instructive to fit the damped operator D within a family of operators D_μ for which an analogue of (1.3) holds:

THEOREM 2. *Assume the same hypotheses for the phase function $S(x, y)$ and the amplitude $\chi(x, y)$ as in Theorem 1, and that the support of χ is a sufficiently small neighborhood of the origin. Set*

$$D_\mu f(x)=\int_{-\infty}^{\infty} e^{i\lambda S(x,\,y)}\,|S''_{xy}(x, y)|^{\mu}\,\chi(x, y)\,f(y)\,dy, \qquad f\in C_0^\infty(\mathbf{R}). \tag{1.4}$$

Then we have

$$\|D_\mu\| \leqslant C\,|\lambda|^{-(\delta+2\mu(1-\delta))/2} \tag{1.5}$$

for μ in the range

$$-\frac{1}{2}\frac{\delta}{1-\delta}<\mu\leqslant\frac{1}{2}. \tag{1.6}$$

The case of the end-point $\mu = -\delta/2(1-\delta)$ is somewhat different from the other cases. Our treatment of this case makes use neither of the oscillating factor $e^{i\lambda S(x,\, y)}$, nor of the fact that the damping factor is of the form $S''_{xy}(x, y)$ for some function $S(x, y)$. Thus it is more appropriate to state it for a general operator of the form

$$Ef(x) = \int_I |\Psi(x, y)|^{-\mu} f(y)\, dy, \tag{1.7}$$

where I is a bounded interval near the origin.

THEOREM 3. *Assume that the function $\Psi(x, y)$ is a real-analytic function in a neighborhood of the origin in $\mathbf{R}^2$. Let δ_0 be the decay rate corresponding to the Newton polyhedron of Ψ. Then for I small enough, the operator E is a bounded operator on $L^2(I)$ for $\mu < 1/2\, \delta_0$. It is also bounded for*

$$\mu = \tfrac{1}{2}\delta_0, \tag{1.8}$$

except possibly when the main face

(A) *reduces to a point vertex*;

(B) *or is parallel to either one of the two axes*;

(C) *or is given by an equation of the form $p + q =$ constant.*

We observe that Theorem 3 is formulated in terms of the usual Newton polyhedron instead of the reduced one. For $\Psi = S''_{xy}$, the value of δ_0 for Ψ works out to be $\delta/1-\delta$, where δ is the Newton decay rate for $S(x, y)$.

Our approach to all three operators D, D_μ, and E, follows the one introduced in [5] for T_λ (and in [4] for some simpler models). The case of the operators D_μ, for $-\frac{1}{2}\delta/1-\delta < \mu < \frac{1}{2}$, is a straightforward adaptation of the arguments given in [5] for T_λ (see Section V below). In particular, the operator D_μ is decomposed into smaller pieces reflecting the proximity to the singular variety $Z = \{(x, y);\, S''_{xy} = 0\}$. The sharp decay rate is strictly smaller than $\frac{1}{2}$ (for degenerate phase functions), and can be derived by an absolute summation of a balance between oscillatory and size estimates. However, the end-points pose new problems. For the upper end-point, corresponding to $\mu = \frac{1}{2}$ and $D_\mu = D$, each of the summands decays already at the desired rate $|\lambda|^{-1/2}$. There is no room left for an absolute summation, and the almost-orthogonality of the summands plays here a primordial role. This situation is in a sense the exact opposite of the one for the lower end-point, corresponding to $\mu = -\frac{1}{2}\delta/1-\delta$ and the more general operator E. Here orthogonality plays no role. Rather, the (x, y) space is directly divided into sectors, over each of which bounds of the Hilbert integral type apply.

II. STATIONARY PHASE LEMMAS WITH UNIFORM BOUNDS

In this section, we derive some estimates we need for oscillatory integral operators. Their key feature is their uniformity with respect to several parameters affecting the phase and the amplitude of the operators.

It is convenient to introduce the notion of *polynomial-like* functions on a finite interval I_0 (with constants C), as smooth real functions $F(x)$ on I_0 satisfying the inequality

$$\sup_{x \in I^*} |F^{(k)}(x)| \leqslant C\, \delta^{-k} \sup_{x \in I} |F(x)| \tag{2.1}$$

for all sub-intervals I of I_0 of length δ, and for $k = 0, 1, 2$. Here I^* is the double of I in I_0. As we saw in [5], functions of polynomial type N, defined in [5] as C^N functions satisfying the following inequality on their N-th derivatives

$$\inf_{x \in I_0} |F^{(N)}(x)| \geqslant c \sup_{x \in I_0} |F^{(N)}(x)|$$

are polynomial-like. Furthermore, the products of polynomial-like functions by smooth and uniformly bounded functions on I_0 are also polynomial-like, with uniform bounds C. It is also easy to verify the following:

Remark. If F is polynomial-like, and $\mu \leqslant |F| \leqslant C\mu$ on I, then

$$\sup_I |\partial_y^k\, |F|^{1/2}| \leqslant C\mu^{1/2}\, |I|^{-k}.$$

We consider operators T_λ on $L^2(\mathbf{R})$, defined as in (1.1), with $S(x, y)$ a smooth and real phase function, and a $C_0^2(\mathbf{R} \times \mathbf{R})$ amplitude $\chi(x, y)$. We shall, for the sake of brevity, say that an operator is supported in some region if its amplitude is supported in that region. We shall make the following assumptions:

(A1) The operator T_λ is supported in a box $\mathscr{B}$ which is ribbon-like (or is a "curved box" in the terminology of [5]), i.e.,

$$\operatorname{supp} \chi \subset \mathscr{B}$$

with $\mathscr{B}$ of the form

$$\mathscr{B} = \{(x, y);\, \phi(x) < y < \phi(x) + \delta,\, \alpha < x < \beta\}.$$

Furthermore,

$$|\partial_y^k \chi(x, y)| \leqslant A\delta^{-k}, \qquad k=0, 1, 2. \tag{2.2}$$

We shall refer to δ as the y-width of $\mathscr{B}$.

(A2) We define the "double" $\mathscr{B}^*$ of $\mathscr{B}$ as the ribbon-like box given by

$$\mathscr{B}^* = \{(x, y); \phi(x) - \delta < y < \phi(x) + 2\delta, \alpha < x < \beta\}. \tag{2.3}$$

(We note that the y-dimension of $\mathscr{B}$ is expanded by a factor of 3, and the x-dimension is not expanded.) Then $S''_{xy}(x, y)$ is polynomial-like in y in $\mathscr{B}^*$ (uniformly in the parameter x), does not change sign in $\mathscr{B}^*$, and satisfies the bounds

$$\begin{aligned} \mu &\leqslant \min_{\mathscr{B}} |S''_{xy}| \\ A_\mu &\geqslant \max_{\mathscr{B}^*} |S''_{xy}(x, y)|. \end{aligned} \tag{2.4}$$

(A3) The function $\phi(x)$ is an increasing function.

Lemma 1. *Under the above assumptions* (A1-3), *we have*

$$\|T_\lambda\| \leqslant A\left(\max \sum_{k=0}^{2} \delta^k \, |\partial_y^k \chi|\right)(\lambda\mu)^{-1/2}. \tag{2.5}$$

The constant A is independent of (α, β), ϕ, δ, χ *and* μ.

We note that Lemma 1 already appears in [5]. However, we have presented here a different proof, which has the advantage of extending to the subsequent almost-orthogonal situation.

To prove Lemma 1, we need two facts which can be verified by inspection.

Fact 1. Let I_{x_1} be the projection on the y-axis of the segment $x=x_1$ intersected with $\mathscr{B}$, i.e. $I_{x_1} = \{y; (x_1, y) \in \mathscr{B}\}$. If $I_{x_1} \cap I_{x_2} \neq 0$, and if we set

$$\mathscr{R} = \{(x, y); x_1 \leqslant x \leqslant x_2, y \in I_{x_1} \cap I_{x_2}\},$$

then $\mathscr{R} \subset \mathscr{B}$. This follows immediately from the monotonicity of ϕ (assumption (A3)).

We can now prove Lemma 1. Let $K(x, y)$ be the kernel of $T_\lambda T_\lambda^*$. Then

$$K(x_1, x_2) = \int_{-\infty}^{\infty} e^{i\lambda(S(x_1, y) - S(x_2, y))} \chi(x_1, y) \overline{\chi(x_2, y)} \, dy. \tag{2.6}$$

On the support of $\chi(x_1, y)\,\overline{\chi(x_2, y)}$, $y \in I_{x_1} \cap I_{x_2}$, and thus

$$S'_y(x_1, y) - S'_y(x_2, y) = \int_{x_1}^{x_2} S''_{uy}(u, y)\,\mathrm{d}u,$$

where, by Fact 1, the relevant points (u, y) in the above integral belong to $\mathscr{B}$. Hence if we write

$$\Phi(y) = S'_y(x_1, y) - S'_y(x_2, y) = \int_{x_1}^{x_2} S''_{uy}(u, y)\,\mathrm{d}u \tag{2.7}$$

we obtain

$$\mu\,|x_1 - x_2| \leqslant |\Phi(y)| \leqslant A\mu\,|x_1 - x_2| \tag{2.8}$$

as a consequence of (A2). In view of the fact that S''_{xy} is polynomial-like in y on the ribbon-like box $\mathscr{B}^*$, we also have

$$|\Phi^{(k)}(y)| \leqslant A\mu\,|x_1 - x_2|\,\delta^{-k}, \qquad y \in I_{x_1} \cap I_{x_2}. \tag{2.9}$$

We then use (2.8) and (2.9) together with two integrations by parts in (2.6). Combining this with assumption (A1) (as in the proof of Lemma 1.1 in [4]) yields

$$|K(x_1, x_2)| \leqslant A\delta\left(\max \sum_{k=0}^{2} \delta^k\,|\partial_y^k \chi|\right)^2 (1 + \lambda^2\mu^2\,\delta^2\,|x_1 - x_2|^2)^{-1}$$

from which

$$\int_{-\infty}^{\infty} |K(x_1, x_2)|\,\mathrm{d}x_i \leqslant A\left(\max \sum_{k=0}^{2} \delta^k\,|\partial_y^k \chi|\right)^2 (|\lambda|\mu)^{-1}.$$

Lemma 1 follows.

In practice, we shall rather need in this paper the following version of Lemma 1:

LEMMA 2. *Under the above assumptions* (A1-3), *the operator* D_λ *defined as in* (1.2) *is bounded on* $L^2(\mathbf{R})$, *with bound*

$$\|D_\lambda\| \leqslant A\left(\max \sum_{k=0}^{2} \delta^k\,|\partial_y^k \chi|\right) |\lambda|^{-1/2}, \tag{2.10}$$

where A *is independent of the box* $\mathscr{B}$, *the width* δ, *and* μ.

In fact, we can apply Lemma 1 with $\chi(x, y)$ replaced by $\tilde{\chi}(x, y) = |S''_{xy}(x, y)|^{1/2}\,\chi(x, y)$. Since $S''_{xy}(x, y)$ is uniformly polynomial-like in y, in view of the remark following (2.1), the new amplitude $\tilde{\chi}(x, y)$ still satisfies the assumption (A1), with however the norm $(\max \sum_{k=0}^{2} \delta^k \, |\partial_y^k \chi|)$ in (2.2) multiplied by at most $\mu^{1/2}$. The inequality (2.10) follows.

We will also need two separate lemmas dealing with almost-orthogonal oscillatory integral operators $T_\lambda(\mathscr{B}_j)$

$$T_\lambda(\mathscr{B}_j)\, f(x) = \int_{-\infty}^{\infty} e^{i\lambda S(x,\, y)} \chi_j(x, y)\, f(y)\, \mathrm{d}y, \qquad j = 1, 2.$$

The first is when the smaller box $\mathscr{B}_2$ is displaced vertically from the larger box $\mathscr{B}_1$, and both are ribbon-like; the second, when the displacement is horizontal, and both have sides parallel to the axes.

In the first lemma, our assumptions are as follows:

(B1) The amplitudes $\chi_j(x, y)$ are supported in two ribbon-like boxes $\mathscr{B}_1$ and $\mathscr{B}_2$ of y-widths δ_1 and δ_2

$$\begin{aligned} \mathscr{B}_1 &= \{(x, y);\, \phi(x) < y < \phi(x) + \delta_1,\, \alpha < x < \beta\} \\ \mathscr{B}_2 &= \{(x, y);\, \psi(x) < y < \psi(x) + \delta_2,\, \alpha < x < \beta\} \end{aligned} \tag{2.11}$$

and satisfy the estimates

$$|\partial_y^k \chi_j(x, y)| \leqslant A\, \delta_j^{-k}, \qquad k = 0, 1, 2, \quad j = 1, 2. \tag{2.12}$$

(B2) we define the double $\mathscr{B}_1^*$ as in (2.3). Furthermore, we also require a very slightly enlarged version $\tilde{\mathscr{B}}_1$ of the box $\mathscr{B}_1$,

$$\mathscr{B}_1 \subset \tilde{\mathscr{B}}_1 \subset \mathscr{B}_1^*$$

defined as

$$\tilde{\mathscr{B}}_1 = \{(x, y);\, \phi(x) - \tfrac{1}{10}\delta_1 < y < \phi(x) + \tfrac{11}{10}\delta,\, \alpha < x < \beta\}. \tag{2.13}$$

Then $S''_{xy}(x, y)$ is polynomial-like in $\mathscr{B}_1^*$, uniformly in x, does not change sign there, and satisfies the bounds

$$\begin{aligned} \mu &\leqslant \min_{\tilde{\mathscr{B}}_1} |S''_{xy}| \\ A\mu &\geqslant \max_{\mathscr{B}_1^*} |S''_{xy}(x, y)|. \end{aligned} \tag{2.14}$$

(B3) The function ϕ is monotonic (say increasing), and

$$\min_{[\alpha,\, \beta]} (\phi'(x)) \geqslant \tfrac{1}{2} \max_{[\alpha,\, \beta]} (\phi'(x)). \tag{2.15}$$

(We do not assume ψ to be monotonic.)

(B4) The box $\mathscr{B}_1$ is the "major" box, and the box $\mathscr{B}_2$ is the "minor" box, in the sense that

$$\mathscr{B}_2 \subset \mathscr{B}_1^*, \qquad \delta_2 \leqslant \delta_1. \tag{2.16}$$

LEMMA 3. *Under the assumptions* (B1-4), *we have*

$$\begin{aligned} \|T_\lambda(\mathscr{B}_1)\, T_\lambda(\mathscr{B}_2)^*\| &\leqslant A(|\lambda|\,\mu)^{-1} \prod_{j=1}^{2} \left(\max \sum_{k=0}^{2} \delta_j^k\, |\partial_y^k \chi_j|\right) \\ \|T_\lambda(\mathscr{B}_2)\, T_\lambda(\mathscr{B}_2)^*\| &\leqslant A(|\lambda|\,\mu)^{-1} \prod_{j=1}^{2} \left(\max \sum_{k=0}^{2} \delta_j^k\, |\partial_y^k \chi_j|\right), \end{aligned} \tag{2.17}$$

where A is again independent of (α, β), ϕ, ψ, δ_j, χ_j, *and* μ.

For the proof we need two further observations.

Fact 2. Suppose $\mathscr{B}_1$ and $\mathscr{B}_2$ are as above. Let L be a line segment, parallel to the x-axis, joining a point $(x, y) \in \mathscr{B}_1$ with a point $(x_2, y) \in \mathscr{B}_2$. Let $\tilde{L} = L \cap \tilde{\mathscr{B}}_1$, where $\tilde{\mathscr{B}}_1$ is defined as previously. Then

$$|\tilde{L}| \geqslant c\,|L|.$$

(With the constant 1/10 in the definition of $\tilde{\mathscr{B}}_1$, and the constant 1/2 in (2.15), we get the constant 1/60 in the above inequality.)

Fact 3. If L is as in Fact 2 above, there exists a rectangle $\mathscr{R}$ of y-width δ_1 so that $L \subset \mathscr{R} \subset \mathscr{B}_1^*$.

Proof of Fact 2. Since $(x, y) \in \mathscr{B}_1$, $0 \leqslant y - \phi(x_1) \leqslant \delta_1$. Also $(x_2, y) \in \mathscr{B}_2 \subset \mathscr{B}_1^*$ implies $-\delta_1 \leqslant y - \phi(x_1) \leqslant 2\delta_1$. Therefore $|\phi(x_1) - \phi(x_2)| \leqslant 3\delta_1$. Set

$$m = \min(\phi'(x)).$$

It follows that $|x_1 - x_2| \leqslant 3\delta_1/m$. Now if (x_2, y) is contained in $\mathscr{B}_1$, then the whole interval joining (x_1, y) to (x_2, y) is contained in $\tilde{\mathscr{B}}_1$, (this is Fact 1, applied to $\tilde{\mathscr{B}}_1 = \mathscr{B}$, since $(x_1, y) \in \tilde{\mathscr{B}}_1$). Thus $L = \tilde{L}$ in this case, and we are done when $(x_2, y) \in \tilde{\mathscr{B}}_1$.

Assume therefore $(x_2, y) \notin \tilde{\mathscr{B}}_1$. Moving from (x_1, y) to (x_2, y) along the line segment joining them, let $(\tilde{x}_1, y)$ be a point on the boundary of $\tilde{\mathscr{B}}_1$. So either $y - \phi(\tilde{x}_1) = \frac{11}{10}\delta_1$ or $y - \phi(\tilde{x}_1) = -\frac{1}{10}\delta_1$. In either case, $|\phi(x_1) - \phi(\tilde{x}_1)| \geqslant \frac{1}{10}\delta_1$, and hence $|x_1 - \tilde{x}_1| \geqslant \delta_1/20m$, since $|\phi'(x)| \leqslant 2m$. Thus, as we have seen, $|L| \leqslant 3\delta_1/m$, while $|\tilde{L}| \geqslant \delta_1/20m$. As a result, $|\tilde{L}| \geqslant c|L|$, with $c = \frac{1}{60}$. We note that the result is independent of $m = \min(\phi'(x))$.

Proof of Fact 3. Assume $x_2 \leqslant x_1$. Since $(x_1, y) \in \mathscr{B}_1$, $y \geqslant \phi(x_1)$, and since ϕ is increasing, $y \geqslant \phi(x)$, whenever $x_2 \leqslant x \leqslant x_1$. Now $(x_2, y) \in \mathscr{B}_2 \subset \mathscr{B}_1^*$, and so $y \leqslant \phi(x_2) + 2\delta_1$. As a result, $y \leqslant \phi(x) + 2\delta_1$, for $x_2 \leqslant x \leqslant x_1$. This means that for the rectangle $\mathscr{R} = [x_2, x_1] \times [y - \delta_1, y] \subset \mathscr{B}_1^*$, and $L = [x_1, x_1] \times \{y\} \subset \mathscr{R}$. A similar argument works for $x_2 \geqslant x_1$.

We turn to the proof of Lemma 3. Let $K(x, y)$ be the kernel of $T_\lambda(\mathscr{B}_1)\, T_\lambda(\mathscr{B}_2)^*$. Then

$$K(x_1, x_2) = \int_{-\infty}^{\infty} e^{i\lambda(S(x_1, y) - S(x_2, y))} \chi_1(x_1, y)\, \overline{\chi_2(x_2, y)}\, dy.$$

If we set again

$$\Phi(y) = S'_y(x_1, y) - S'_y(x_2, y),$$

then on the support of $\chi_1(x_1, y)\, \overline{\chi_2(x_2, y)}$, we have

$$c\mu\, |x_1 - x_2| \leqslant |\Phi(y)| \leqslant A\mu\, |x_1 - x_2|. \tag{2.18}$$

In fact, $\Phi(y) = \int_{x_1}^{x_2} S''_{uy}(u, y)\, du$, and for the relevant points $((x_1, y) \in \mathscr{B}_1$, $(x_2, y) \in \mathscr{B}_2)$, the segment $L = \{(u, y)\}$ belongs to $\mathscr{B}_1^*$, where S'' does not change sign (see (B2) and Fact 3). Moreover, by Fact 2, for a positive fraction of the length of L, i.e. on $\tilde{L}$, we have

$$|S''_{uy}(u, y)| \geqslant \mu, \tag{2.19}$$

by the first inequality of assumption (B2). With this, the left hand inequality in (2.18) is established. The right inequality follows in view of the second inequality in (B2).

We also have

$$|\Phi^{(k)}(y)| \leqslant A\mu\, \delta_1^{-k}\, |x_1 - x_2| \tag{2.20}$$

if we use Fact 3.

Once we have (2.18) and (2.20), we may integrate by parts in the integral defining the kernel $K(x_1, x_2)$ and obtain once again

$$|K(x_1, x_2)| \leqslant A\, \delta_2 \prod_{j=1}^{2} \left(\max \sum_{k=0}^{2} \delta_j^k\, |\partial_y^k \chi_j| \right) (1 + \lambda^2 \mu^2\, \delta_2^2\, |x_1 - x_2|^2)^{-1}$$

which proves the first inequality in (2.17). The inequality for $\| T_\lambda(\mathscr{B}_2)\, T_\lambda(\mathscr{B}_1)^* \|$ follows by taking adjoints.

In the second almost-orthogonality lemma, we shall be dealing again with two boxes $\mathscr{B}_1$ and $\mathscr{B}_2$, but this time both would be rectangular boxes with sides parallel to the axes; moreover, the minor box $\mathscr{B}_2$ will be contained in a horizontal translate of the major box $\mathscr{B}_1$. The precise definitions and assumptions are as follows:

$$
\begin{aligned}
&\mathscr{B}_1 = \{(x, y); a_1 < y < b_1, \alpha_1 < x < \beta_1\}, b_1 - a_1 = \delta_1 \\
&\tilde{\mathscr{B}}_1 = \{(x, y); a_1 < y < b_1, \tilde{\alpha} < x < \tilde{\beta}\} \\
&\tilde{\alpha}_1 = \alpha_1 - \tfrac{1}{10}(\beta_1 - \alpha_1), \tilde{\beta}_1 = \beta_1 + \tfrac{1}{10}(\beta_1 - \alpha_1) \\
&\mathscr{B}_1^* = \{(x, y); a_1 < y < b_1, \alpha^* < x < \beta^*\} \\
&\alpha_1^* = \alpha_1 - (\beta_1 - \alpha_1), \beta_1^* = \beta_1 + (\beta_1 - \alpha_1).
\end{aligned}
\tag{2.21}
$$

Note that the expansions are in the x-dimension only.

We shall also have another "minor" box $\mathscr{B}_2$

$$
\mathscr{B}_2 = \{(x, y); a_2 < y < b_2, \alpha_2 < x < \beta_2\}, \qquad b_2 - a_2 = \delta_2.
$$

We define the operators $T_\lambda(\mathscr{B}_j)$ by the same expressions as earlier and make the following assumptions:

(C1) χ_j is supported in $\mathscr{B}_j$, and $|\partial_y^k \chi_j(x, y)| \leqslant A\, \delta_j^{-k}$, $k = 0, 1, 2$.

(C2) We define the span $span(\mathscr{B}_1, \mathscr{B}_2)$, as the union of all line segments parallel to the x-axis, which joins a point $(x_1, y) \in \mathscr{B}_1$ with a point $(x_2, y) \in \mathscr{B}_2$. Then $S''_{xy}(x, y)$ is polynomial-like in y, uniformly in x in $\mathscr{B}_1^*$, does not change sign in the span $span(\mathscr{B}_1, \mathscr{B}_2)$, and satisfies the bounds

$$
\begin{aligned}
\mu &\leqslant \min_{\tilde{\mathscr{B}}_1} |S''_{xy}| \\
A\mu &\geqslant \max_{span(\mathscr{B}_1, \mathscr{B}_2)} |S''_{xy}|.
\end{aligned}
\tag{2.22}
$$

(C3) $\mathscr{B}_2 \subset \mathscr{B}_1^*$. Note that this implies that $\delta_2 \leqslant \delta_1$.

We would like to point out that in applications, the function S''_{xy} will usually change sign in the double of the larger box $\mathscr{B}_1$. Thus it is crucial that Lemmas 4 and 5 below be applicable under the less restrictive assumption that S''_{xy} not change sign only in the span $span(\mathscr{B}_1, \mathscr{B}_2)$ of $\mathscr{B}_1$ and $\mathscr{B}_2$.

LEMMA 4. *Under the preceding assumptions* (C1–3), *we have*

$$
\begin{aligned}
\|T_\lambda(\mathscr{B}_1)\, T_\lambda(\mathscr{B}_2)^*\| &\leqslant A(\lambda\mu)^{-1} \prod_{j=1}^{2} \left(\max \sum_{k=0}^{2} \delta_j^k\, |\partial^k \chi_j| \right) \\
\|T_\lambda(\mathscr{B}_2)\, T_\lambda(\mathscr{B}_1)^*\| &\leqslant A(\lambda\mu)^{-1} \prod_{j=1}^{2} \left(\max \sum_{k=0}^{2} \delta_j^k\, |\partial^k \chi_j| \right),
\end{aligned}
\tag{2.23}
$$

where A is independent of the boxes $\mathscr{B}_1, \mathscr{B}_2$, of the amplitudes χ_j, and of μ.

The proof depends on the following variants of Facts 2 and 3 above.

Fact 2′. Let L be a line segment parallel to the x axis, joining a point in $\mathscr{B}_1$ with a point in $\mathscr{B}_2$. Let $\tilde{L} = L \cap \tilde{\mathscr{B}}_1$. Then $|\tilde{L}| \geqslant c\,|L|$.

Proof of Fact 2′. Let $(x_1, y) \in \mathscr{B}_1$, and $(x_2, y) \in \mathscr{B}_2$, and L the segment joining them. If $(x_2, y) \in \tilde{\mathscr{B}}_1$, then $\tilde{L} = L$, and we are done. Otherwise, $|\tilde{L}| \geqslant \frac{1}{10}(\beta - \alpha)$, while in any case $|L| \leqslant 3(\beta - \alpha)$ (since $L \subset \mathscr{B}_1^*$). This proves the assertion with $c = \frac{1}{30}$.

Fact 3′. If L is as in Fact 2′ above, there is a rectangle $\mathscr{R}$ of y-width δ_2, so that $L \subset \mathscr{R} \subset span(\mathscr{B}_1, \mathscr{B}_2)$. Here we note that δ_2 is used instead of δ_1 as in Fact 3.

In fact, it suffices to take $\mathscr{R} = span(\mathscr{B}_1, \mathscr{B}_2)$.

To prove Lemma 4, we define $\Phi(y)$ as before by the same integral formula (2.6). Now on the support of $\chi_1(x_1, y)\,\overline{\chi_2(x_2, y)}$, the points (u, y) in the integrand defining $\Phi(y)$, belong to $span(\mathscr{B}_1, \mathscr{B}_2)$. Therefore by the assumption (C2), together with Fact 2′, we obtain

$$c\mu\,|x_1 - x_2| \leqslant |\Phi(y)| \leqslant A\mu\,|x_1 - x_2|. \tag{2.24}$$

Now if we apply (2.1) to the above, keeping in mind Fact 3′, we obtain

$$|\Phi^{(k)}(y)| \leqslant A\mu\,\delta_2^{-k}\,|x_1 - x_2| \tag{2.25}$$

(observe that, as opposed to (2.20), we have the factor δ_2^{-k} instead of δ_1^{-k}). Using (2.24) and (2.25), we can now use the standard integration by parts argument to find

$$|K(x_1, x_2)| \leqslant A \prod_{j=1}^{2} \left(\max \sum_{k=0}^{2} \delta_j^k\,|\partial_y^k \chi_j| \right) \delta_2 (1 + \lambda^2 \mu^2 \delta_2^2\,|x_1 - x_2|^2)^{-1}$$

which then proves Lemma 4.

Again, in this paper, we rather need the following version of Lemmas 3 and 4:

Lemma 5. *Let $D(\mathscr{B}_j)$ be operators defined as in* (1.2) *by*

$$D(\mathscr{B}_j)\,f(x) = \int_{-\infty}^{\infty} e^{i\lambda S(x, y)}\,|S''_{xy}(x, y)|^{1/2}\,\chi_j(x, y)\,f(y)\,dy, \quad f \in C_0^\infty(\mathbf{R}). \tag{2.26}$$

Then under either set of assumptions (B1-4), *or* (C1-3), *we have*

$$\|D(\mathscr{B}_1)\, D(\mathscr{B}_2)^*\| \leqslant A \prod_{j=1}^{2} \left(\max \sum_{k=0}^{2} \delta_j^k\, |\partial_y^k \chi_j| \right) \left(\frac{\sup_{\mathscr{B}_2} |S''_{xy}|}{\sup_{\mathscr{B}_1} |S''_{xy}|} \right)^{1/2} |\lambda|^{-1}$$

$$\|D(\mathscr{B}_2)\, D(\mathscr{B}_1)^*\| \leqslant A \prod_{j=1}^{2} \left(\max \sum_{k=0}^{2} \delta_j^k\, |\partial_y^k \chi_j| \right) \left(\frac{\sup_{\mathscr{B}_2} |S''_{xy}|}{\sup_{\mathscr{B}_1} |S''_{xy}|} \right)^{1/2} |\lambda|^{-1} \tag{2.27}$$

In fact, as in the proof of Lemma 2, the expressions $|S''_{xy}(x, y)|^{1/2}\, \chi_j(x, y)$ define new amplitudes verifying the analogues of (B1) and (C1), with the norm $\max \sum_{k=0}^{2} \delta_j^k\, |\partial_y^k \chi_j|$ dilated at most by $\sup_{\mathscr{B}_i} |S''_{xy}|^{1/2}$, since S''_{xy} is polynomial-like in both $\mathscr{B}_1$ and $\mathscr{B}_2$. These new factors modify the estimate (2.23) to the desired estimate (2.27).

In subsequent applications, it is convenient to note the following:

- A similar statement holds for D^*D, if we interchange the roles of x and y.
- Actually, in the definition of $\tilde{\mathscr{B}}_1$, the fraction $\frac{1}{10}$ can be replaced by any small positive constant. Similarly, in the definition of $\mathscr{B}_1^*$, other constants rather than 1 in front of δ_1 will do equally well. The choice of other constants of dilations will just lead to other constants in the estimates for the operators $T_\lambda(\mathscr{B}_j)$ and $D(\mathscr{B}_j)$, constants which remain bounded as long as $\tilde{\mathscr{B}}_1$ does not approach $\mathscr{B}_1$ arbitrarily closely, and the y-width of $\mathscr{B}_1^*$ does not become arbitrarily large.
- The same applies to the constant $\frac{1}{2}$ in the assumption (B3).
- When the context makes it evident, we shall abbreviate both conditions (2.14) and (2.22) by saying that

$$|S''_{xy}| \sim \mu \qquad \text{on } \mathscr{B}_1. \tag{2.28}$$

- Henceforth, in presence of boxes $\mathscr{B}_i$ satisfying conditions of either form (B1–4) or (C1–3), we shall refer to the expression $(\sup_{\mathscr{B}_2} |S''_{xy}| / \sup_{\mathscr{B}_1} |S''_{xy}|)^{1/2}$ as the "small box/big box ratio," and often denote it by

$$\left| \frac{S''_{small}}{S''_{big}} \right|^{1/2}. \tag{2.29}$$

III. A MODEL CASE

As in our earlier paper [5], it is instructive to begin with a careful treatment of a typical case, which does not require the heavy notation of successive resolutions. Thus we consider the case

$$S''(x, y) = (y - x^a)(y - x^a - x^A)(y - x^b), \tag{3.1}$$

where $0 < a < b$, $a < A$. Let $\chi \in C_0^\infty(2^{-1}, 2^2)$ be a function satisfying

$$\sum_{m=-\infty}^{\infty} \chi(2^m x) = 1, \qquad 0 < x < \infty.$$

Then the operator D can be decomposed as

$$D = \sum_{\sigma, \tau = \pm} \sum_{-\infty < j, k < \infty} U_{jk}^{\sigma\tau} \tag{3.2}$$

with each component $U_{jk}^{\sigma\tau}$ defined by

$$U_{jk}^{\sigma\tau} f(x) = \int_{-\infty}^{\infty} e^{i\lambda S(x, y)} \chi(\sigma 2^j x) \chi(\tau 2^k y) \, |S''_{xy}(x, y)|^{1/2} \, \chi(x, y) \, f(y) \, dy. \tag{3.3}$$

We can assume, without loss of generality, that the support of $\chi(x, y)$ is a neighborhood of the origin, so small that

(i) the only non-vanishing $U_{jk}^{\sigma\tau}$ of the above decomposition occur only for j, k very large;

(ii) in each quadrant, restricted to the support of $\chi(x, y)$, the functions $y = x^a$, $y = x^a + x^A$, $y = x^b$ are all strictly monotone;

(iii) in each dyadic interval I_0 of the form $2^{-m-1} \leqslant x \leqslant 2^{-m+2}$, the derivatives of each of the roots $\phi(x) = x^a$, $\phi(x) = x^a + x^A$, $\phi(x) = x^b$, are of constant size. In particular, the condition

$$\min_{I_0} \phi' \geqslant c \max_{I_0} \phi'$$

is satisfied with a constant c independent of j.

Since S''_{xy} is a monic polynomial in y in this example, S''_{xy} is a fortiori polynomial-like.

We shall consider separately the contributions of the operators $U_{jk}^{\sigma\tau}$ in the ranges $k \ll aj$, $aj \sim k$, $aj \ll k \ll bj$, $bj \sim k$, $bj \ll k$. Here the notations $k \ll aj$ and $k \sim aj$ for example indicate respectively the ranges

$$\begin{aligned} k + C &\leqslant aj \\ |k - aj| &\leqslant C \end{aligned} \tag{3.4}$$

for some large positive integer C, depending possibly on the size of the support of $\chi(x, y)$, and the coefficients and exponents occurring in the factorization of S''_{xy}, but not on j and k.

Contribution of the Range $aj \ll k \ll bj$

In this range, we treat separately the sums $\sum U^{\sigma\tau}_{jk}$ for fixed σ and fixed τ. Thus we set, say, $\sigma = \tau = +$, and consider only positive x and positive y. It is immediately seen that if the constant C in (3.4) is chosen large enough, then each U^{++}_{jk} satisfies the conditions of Lemma 2, with the box $\mathscr{B}$ a rectangular box of the form $x \sim 2^{-j}$ and $y \sim 2^{-k}$, and thus

$$\| U^{++}_{jk} \| \leqslant |\lambda|^{-1/2}.$$

To simplify the notation, we have, in this inequality as well as in the rest of the paper, omitted all multiplicative constants which are independent of λ and of summation indices which may occur. Next, $U^{++}_{jk}(U^{++}_{j'k'})^* = 0$ unless the y-supports (i.e. the union of all the supports of the amplitudes, viewed as functions of y for fixed x) of U^{++}_{jk} and $U^{++}_{j'k'}$, intersect. This implies that $|k - k'| \leqslant 3$. We may as well assume that $k = k'$. In this case, the small box/big box ratio (c.f. (2.27) and (2.29)) for two operators U^{++}_{jk} and $U^{++}_{j'k'}$ is given by

$$\left| \frac{S''_{small}}{S''_{big}} \right|^{1/2} \sim 2^{-a\,|j-j'|\,2}. \tag{3.5}$$

Note that S''_{xy} does not change sign in the span of the two boxes, and that the assumptions (C) of Lemma 5 are satisfied. Thus

$$\| U^{++}_{jk} (U^{++}_{j'k'})^* \| \leqslant |\lambda|^{-1/2}\, 2^{-a\,|j-j'|/2}. \tag{3.6}$$

Similar estimates hold for $(U^{++}_{jk})^*\, U^{++}_{j'k'}$, since the hypotheses of Lemma 5 still hold when we reverse the roles of x and y. The desired estimate follows by almost-orthogonality.

Contribution of the Range $bj \ll k$

In this range, we note that $|y|$ can become arbitrarily small. Thus we can consider separately each sign σ for x (say $\sigma = +$), but not the signs $\tau = \pm$ for y. We set then

$$\sum_{bj \ll k,\, \tau = \pm} U^{+\tau}_{jk} = \sum_j \sum_{bj \ll k} (U^{++}_{jk} + U^{+-}_{jk}) \equiv \sum_j U^{+}_{j}. \tag{3.7}$$

Evidently, the amplitude $\chi_j(x, y)$ of U_j^+ is supported in a rectangle $\mathscr{B}_j$ of the form $\{x \sim 2^{-j}\} \times \{|y| \leqslant 2^{-bj}\}$, and satisfies

$$|\partial_y^\alpha \chi_j(x, y)| \leqslant 2^{-bj\alpha}.$$

Furthermore, the conditions (C) of Lemma 5, with $\mathscr{B}_1$ and $\mathscr{B}_2$ there corresponding respectively to $\mathscr{B}_j$ and $\mathscr{B}_{j'}$, $(j < j')$, are all verified, with

$$|S''| \sim 2^{-3aj} \qquad \text{on } \mathscr{B}_j, \qquad |S''| \sim 2^{-3aj'} \qquad \text{on } \mathscr{B}_{j'}. \tag{3.8}$$

Lemma 5 implies then

$$\| U_j^+ (U_{j'}^+)^* \| \leqslant |\lambda|^{-1} 2^{-3a|j-j'|/2}$$

and the almost-orthogonality lemma shows that the operator sum in (3.7) is bounded by $|\lambda|^{-1/2}$.

The region $k \ll aj$ is treated in a similar way, this time by considering separately each τ, and assembling the contributions of $\sigma = +$ with $\sigma = -$.

Contributions of the Range $k \sim bj$

It suffices to treat each (σ, τ) separately, so we select $(\sigma, \tau) = (+, +)$, the other cases being similar. Furthermore, the operators U_{jk}^{++} and $U_{j'k'}^{++}$ are then pairwise orthogonal (by comparing their supports) unless we have both $|k - k'| \leqslant 3$ and $|j - j'| \leqslant 3$. It suffices then to show that $\| U_{jk}^{++} \| \leqslant |\lambda|^{-1/2}$ for each j, with bounds uniform in j.

We introduce a further partition of the box $x \sim 2^{-j}$, $y \sim 2^{-bj}$

$$1 = \sum_{\kappa = \pm} \sum_{m=-\infty}^{\infty} \chi(\kappa 2^m (y - x^b)) \tag{3.9}$$

leading to

$$U_{jk}^{++} = \sum_{\kappa = \pm} \sum_{m=-\infty}^{\infty} U_{jk;m}^{++\kappa}. \tag{3.10}$$

We concentrate on each sign κ separately, say $\kappa = +$. The kernel of $U_{jk;m}^{+++}$ is supported in a box $\mathscr{B}_m$ of the form (2.2), with $\phi(x) = x^b - 2^{-m-1}$, and $\delta = \frac{7}{2} 2^{-m}$. Clearly $|S''_{xy}|$ is bounded from below on a small dilate, and above on a large dilate of $\mathscr{B}_m$, by (multiples of) the same constant $2^{-2aj} 2^{-m}$

$$|S''| \sim 2^{-2aj} 2^{-m} \qquad \text{on } \mathscr{B}_m.$$

For $|m' - m|$ large and, say, $m' > m$, a suitable dilate by a factor less than 4 of the larger box $\mathscr{B}_m$ will contain the smaller box $\mathscr{B}_{m'}$. The conditions

(B1–B4) of Lemma 5 are all satisfied, with the small box/big box ratio for the two operators $U^{+++}_{jk;\,m}$ and $U^{+++}_{jk;\,m'}$, given by $2^{-|m-m'|/2}$. Thus

$$\| U^{+++}_{jk;\,m}(U^{+++}_{j'k';\,m'})^{*}\| \leqslant |\lambda|^{-1/2}\, 2^{-|m-m'|/2}. \tag{3.11}$$

Similar estimates hold for $(U^{+++}_{jk;\,m})^{*}\, U^{+++}_{j'k';\,m'}$, since the supports of these operators still fit inside ribbon-like boxes, after interchanging the roles of x and y. Hence $\| U^{++}_{jk} \| \leqslant |\lambda|^{-1/2}$ by almost-orthogonality.

Contribution of the Range $k \sim aj$

We proceed first in complete analogy with the range $k \sim bj$, i.e., consider operators $U^{+++}_{jk;\,m}$ corresponding to the insertion of a cut-off $\chi(2^{m}(y-x^{a}))$. Since the signs $+++$ no longer play any role, we can ignore them. In view of the presence of two asymptotically equal roots $y-x^{a}$ and $y-x^{a}-x^{A}$, we need to consider in turn several ranges for the index m, namely

$$m \sim aj,\ aj \ll m \ll Aj,\ m \sim Aj,\ Aj \ll m. \tag{3.12}$$

The first range consists only of a boundedly finite number of terms m. By Lemma 2, applied to boxes $\mathscr{B}$ of the form $y-x^{a} \sim 2^{-aj}$, each of these terms is bounded in norm by $|\lambda|^{-1/2}$. Thus their sum is bounded by $|\lambda|^{-1/2}$.

In the second and fourth ranges, we have almost-orthogonality, for operators corresponding to distinct values m and m'. Here we are applying Lemma 5, with boxes of the form $y-x^{a} \sim 2^{-m}$, $y-x^{a} \sim 2^{-m'}$, and the following small box/big box ratios

$$aj \ll m \ll Aj\colon \quad \left|\frac{S''_{small}}{S''_{big}}\right|^{1/2} \sim 2^{-|m-m'|};$$

$$Aj \ll m\colon \quad \left|\frac{S''_{small}}{S''_{big}}\right|^{1/2} \sim 2^{-|m-m'|/2}.$$

In the third range $m \sim Aj$, there are only finitely many terms, and it suffices to establish the bound $|\lambda|^{-1/2}$ for each of them. We need to superpose another partition

$$1 = \sum_{\sigma=\pm} \sum_{M=-\infty}^{\infty} \chi(\sigma 2^{M}(y-x^{a}-x^{A})) \tag{3.13}$$

resulting in a decomposition

$$U_{jk;\,m} = \sum_{\sigma=\pm} \sum_{M=-\infty}^{\infty} U^{\sigma}_{jk;\,mM}.$$

We note that in reality, the range of M is $M \geqslant aj$. We consider first the range $M \gg aj$ for M. In this case, however, it is once again easily seen that Lemma 5 is applicable to operators $U^{\sigma}_{jk;\,mM}$ and $U^{\sigma}_{jk;\,mM'}$ corresponding to different values $M > M' \gg aj$, with the boxes $\mathscr{B}_i$ of the form $y-(x^a+x^A) \sim 2^{-M}$ and $y-(x^a+x^A) \sim 2^{-M'}$. We have then almost-orthogonality, with the small box/big box ratio given by $|S''_{small}/S''_{big}|^{1/2} \sim 2^{-|M-M'|/2}$. Thus these operators also sum at most to $|\lambda|^{-1/2}$.

Finally, we need to consider the range $M \sim aj$, where there are only boundedly many terms $U^{\sigma}_{jk;\,mM}$. To be in position to apply Lemma 2, we introduce a finer partition of a neighborhood of the support of the kernel of $U^{\sigma}_{jk;\,mM}$ by inserting a partition of unity subordinated to a covering of the form

$$\begin{aligned} 2^{-j-1}+p\varepsilon 2^{-j} &< x < 2^{-j-1}+(p+1)\,\varepsilon 2^{-j}, \\ 2^{-aj}+q\varepsilon 2^{-aj} &< y-(x^a+x^A) < 2^{-aj}+(q+1)\,\varepsilon 2^{-aj}, \end{aligned} \tag{3.14}$$

where ε is a small positive number, depending only on the coefficients of S''_{xy}, and both p and q span a large, but finite range of order $O(\varepsilon^{-1})$. We note that each of these "tiny" boxes defined by the above equations satisfy the criteria of Lemma 2. Furthermore, on each box intersecting a ε neighborhood of the support of the kernel of $U_{jk;\,M}$, we have both estimates

$$y-x^a \sim 2^{-aj}, \qquad y-(x^a+x^A) \sim 2^{-aj}$$

at some point, and hence in a dilate of the whole tiny box (3.14). Thus Lemma 2 applies, giving the bound $|\lambda|^{-1/2}$ for each summand. The treatment of the model case is complete.

IV. PROOF OF THEOREM 1

Without loss of generality, we can assume that the factor $\chi(x, y)$ in the amplitude of the operator D is supported in a small neighborhood of the origin in $\mathbf{R}^2$. If the Hessian S''_{xy} does not vanish at the origin, then the decay rate $\|D\| \leqslant |\lambda|^{-1/2}$ is a special case of the well-known theorem of Hörmander [2] on oscillatory integral operators with non-degenerate phases. Otherwise, if $S''_{xy}(0,0)=0$, and S''_{xy} is not identically zero, the Weierstrass preparation theorem shows that $S''_{xy}(x, y)$ can be expressed as

$$S''_{xy}(x, y) = U(x, y)\, x^r y^s \prod_{\nu} (y - r_\nu(x)), \tag{4.1}$$

where $U(x, y)$ is a non-vanishing factor which we shall ignore henceforth, r, s are nonnegative integers, and the non-trivial roots $r_\nu(x)$ are Puiseux series in x which are not identically zero. We organize these factors by the exponents of x occurring in the Puiseux expansions of the roots. Thus, let a_ν be the leading exponent of $r_\nu(x)$, i.e. $r_\nu(x) - c_\nu x^{a_\nu} = O(x^{A_\nu})$, for suitable coefficients c_ν, and exponents $A_\nu > a_\nu$. We index the *distinct* exponents a_ν's by a_l's, and order them in increasing order

$$0 < a_1 < a_2 < \cdots < a_l < a_{l+1} < \cdots < a_n. \tag{4.2}$$

We can then define the cluster of roots with leading exponents a_l by

$$\Phi\begin{bmatrix} \cdot \\ l \end{bmatrix} = \prod_{a_\nu = a_l} (y - r_\nu(x)). \tag{4.3}$$

Next, for fixed l, we need to distinguish between the roots appearing in the factorization (4.3) for $\Phi[{}^{\cdot}_{l}]$, depending on the coefficients of x^{a_l} in their Puiseux expansions. Consider then the set of all *distinct* coefficients c_l^α, and set

$$\begin{aligned} \Phi\begin{bmatrix} \cdot \\ l \end{bmatrix} &= \prod_\alpha \Phi\begin{bmatrix} \alpha \\ l \end{bmatrix} \\ \Phi\begin{bmatrix} \alpha \\ l \end{bmatrix} &= \prod_{r_\nu = c_l^\alpha x_l^\alpha + \cdots} (y - r_\nu(x)). \end{aligned} \tag{4.4}$$

Proceeding in this vein, we arrive at a complete classification of the roots of $S''_{xy}(x, y)$ into

$$\begin{aligned} \Phi\begin{bmatrix} \alpha_1 \cdots \alpha_{N-1} & \cdot \\ l_1 \cdots l_{N-1} & l_N \end{bmatrix} &= \prod_{\alpha_N} \Phi\begin{bmatrix} \alpha_1 \cdots \alpha_N \\ l_1 \cdots l_N \end{bmatrix} \\ \Phi\begin{bmatrix} \alpha_1 \cdots \alpha_N \\ l_1 \cdots l_N \end{bmatrix} &= \prod (y - r_\nu(x)), \end{aligned} \tag{4.5}$$

where the last product in (4.4) is over roots $r_\nu(x)$ with the leading coefficients

$$r_\nu(x) = c_{l_1}^{\alpha_1} x^{a_{l_1}} + c_{l_1 l_2}^{\alpha_1 \alpha_2} x^{a_{l_1 l_2}^{\alpha_1}} + \cdots + c_{l_1 \cdots l_N}^{\alpha_1 \cdots \alpha_N} x^{a_{l_1 \cdots l_{N-1} l_N}^{\alpha_1 \cdots \alpha_{N-1}}} + \cdots.$$

There is a natural partial ordering among all the above clusters of roots, with a cluster being less than another cluster if its roots are contained in the roots of the latter cluster. It is convenient to refer to the integer N in (4.5) as the *order of magnification* of the corresponding clusters

$$\Phi\begin{bmatrix} \alpha_1 \cdots \alpha_N \\ l_1 \cdots l_N \end{bmatrix} \quad \text{and} \quad \Phi\begin{bmatrix} \alpha_1 \cdots \alpha_{N-1} & \cdot \\ l_1 \cdots l_{N-1} & l_N \end{bmatrix}.$$

Clearly, by going sufficiently far in this process, we can arrive at a classification of roots in which all the clusters

$$\Phi\begin{bmatrix}\alpha_1 \cdots \alpha_N \\ l_1 \cdots l_N\end{bmatrix}$$

which are minimal with respect to the above partial ordering, consist only of a single root, repeated according to its multiplicity.

To simplify the notation, we shall not indicate explicitly the (finite) ranges of the various indices occurring in clusters of roots (except for the a_l's, which range from a_1 to a_n, as indicated in (4.2)). Rather, we just need the following "generalized multiplicities"

$$N\begin{bmatrix}\cdot \\ \cdot\end{bmatrix} = \deg \Phi\begin{bmatrix}\cdot \\ \cdot\end{bmatrix}, \tag{4.6}$$

where "deg" indicates the degree in y of the cluster Φ, i.e., the number of roots it contains, including multiplicities.

As in (3.2), we decompose the operator D as a sum of operators $U_{jk}^{\sigma\tau}$. The proof of the theorem consists in showing that this sum consists essentially of almost-orthogonal pieces, all of which satisfy the desired bound $|\lambda|^{-1/2}$. Again, we consider separately the possibilities $a_l \ll k \ll a_{l+1} j$, $k \sim a_l j$ for some j, and $k \ll a_n j$, and $k \ll a_1 j$. Recall that the symbols $\ll$, $\sim$, and $\gg$ are taken in the sense of (3.4).

Contribution of the Range $a_l j \ll k \ll a_{l+1} j$

We note that in this case, the conditions $|y| \sim 2^{-k}$, $|x| \sim 2^{-j}$ imply that the regions with distinct signs for y or x are all disjoint, and can be treated separately. Thus we assume that $\sigma = \tau = +$, and discuss only the case $y \sim 2^{-k}$, $x \sim 2^{-j}$. Now $U_{jk}^{++}(U_{j'k'}^{++})^* = 0$ unless the y-supports of U_{jk}^{++} and $(U_{j'k'}^{++})^*$ intersect, which implies that $|k-k'| \leqslant 3$. Next, on the box $\mathscr{B}_{jk} = y \sim 2^{-k}$, $x \sim 2^{-j}$ which contains the support of U_{jk}^{++}, we have

$$\begin{aligned} \Phi\begin{bmatrix}\cdot \\ \mu\end{bmatrix} &\sim 2^{-ja_\mu N[\dot{\mu}]}, \qquad \mu \leqslant l \\ \Phi\begin{bmatrix}\cdot \\ \mu\end{bmatrix} &\sim 2^{-kN[\dot{\mu}]}, \qquad \mu \geqslant l+1 \end{aligned} \tag{4.7}$$

and hence

$$|S''_{xy}| \sim 2^{-jr} 2^{-ks} 2^{-j\sum_{\mu=1}^{l} a_\mu N[\dot{\mu}]}\, 2^{-k\sum_{\mu=l+1}^{n} N[\dot{\mu}]} \qquad \text{on } \mathscr{B}_{jk}. \tag{4.8}$$

In particular, for $j > j'$, we obtain

$$\left|\frac{S''_{small}}{S''_{big}}\right|^{1/2} \sim 2^{-r(j-j')/2} 2^{-ks} 2^{-(j-j')\sum_{\mu=1}^{l} a_\mu N\left[{}^{\cdot}_{\mu}\right]/2} \tag{4.9}$$

from which it follows by Lemma 5

$$\| U_{jk}^{++} (U_{j'k'}^{++})^* \| \leqslant |\lambda|^{-1}\, 2^{-(j-j')\,r/2} 2^{-(j-j')\sum_{\mu=1}^{l} a_\mu N\left[{}^{\cdot}_{\mu}\right]/2} \tag{4.10}$$

Similarly, we also have

$$\|(U_{jk}^{++})^*\, U_{jk'}^{++}\| \leqslant |\lambda|^{-1}\, 2^{-|k-k'|\,s/2} 2^{-|k-k'|\sum_{\mu=l+1}^{n} N\left[{}^{\cdot}_{\mu}\right]/2}. \tag{4.11}$$

Thus the operators U_{jk}^{++} are almost-orthogonal, and their sum has operator norm bounded by $|\lambda|^{-1/2}$.

Contributions of the Range $k \gg a_n j$

The proof depends on whether or not the factor y^s occurs in the factorization (4.1). First, we consider the case where it does occur, with exponent $s > 0$. In this case, we may fix a choice of signs, say $\sigma = \tau = +$ to be specific, and use an identical argument to the previous one for the range $a_l\, j \ll k \ll a_{l+1} j$, with formally $l = n$. Thus $U_{jk}^{++}(U_{j'k'}^{++})^* = 0$ unless $|k - k'| \leqslant 3$, and $\| U_{jk}^{++}(U_{j'k'}^{++})^* \|$ can be bounded as in (4.10) with $l = n$. On the other hand, $(U_{jk}^{++})^* U_{j'k'}^{++} = 0$ unless $|j - j'| \leqslant 3$, and $\|(U_{jk}^{++})^*\, U_{j'k'}^{++})\|$ can be bounded as in (4.11). For $s > 0$, we obtain an almost-orthogonal sum just as before.

Next, we consider the case where $s = 0$. This means that $y = 0$ is not a root, and in a sense, the division into regions $y > 0$ and $y < 0$ is an artifact of our decomposition. Thus we treat separately the ranges $\sigma = +$ and $\sigma = -$, but incorporate the two ranges $\tau = +$ and $\tau = -$. Set then $\sigma = +$, and

$$U_j^{+} = \sum_{k \gg a_n j}\ \sum_{\tau = \pm} U_{jk}^{+\tau}. \tag{4.12}$$

Then U_j^{+} has amplitude $|S''_{xy}(x, y)|^{1/2}\, \chi_j(x, y)$ supported in the rectangular box $\mathscr{B}_j = \{x \sim 2^{-j}\} \times \{|y| \leqslant 2^{-a_n j}\}$, satisfying bounds of the form

$$|\partial_y^\alpha\, \chi_j(x, y)| \leqslant 2^{\alpha a_n j}. \tag{4.13}$$

Furthermore, on the support $\mathscr{B}_j$ of U_j^{+}, we have

$$|S''| \sim 2^{-jr} 2^{-j\sum_{l=1}^{n} a_l N\left[{}^{\cdot}_{l}\right]}. \tag{4.14}$$

Lemma 5 under the conditions (C1–C3) applies, giving

$$\|U_j^+(U_{j'}^+)^*\| \; 2^{-|j-j'|\, r/2} 2^{-|j-j'| \sum_{l=1}^n a_l N \left[{\cdot \atop l}\right]/2} \, |\lambda|^{-1}. \tag{4.15}$$

Since $(U_j)^* \, U_{j'}^+ = 0$ for $|j-j'| \leqslant 3$, we have almost-orthogonality, and thus $\|\sum_j U_j\| \leqslant |\lambda|^{-1/2}$.

The range $k \ll a_1 j$ can be treated in a similar way (this time by distinguishing the cases of $r=0$ and $r>0$, and in the latter case, gathering all operators $U_{jk}^{\sigma\tau}$ for both signs of σ). Thus we turn to the:

Contributions of the Range $k \sim a_l j$

Here it suffices to show, by orthogonality, that each of the operators $U_{jk}^{\sigma\tau}$ is bounded by $|\lambda|^{-1/2}$. We can consider separately each choice of signs $\pm$ for σ, τ, say $\sigma = \tau = +$, so that both x and y are positive. Henceforth, we drop the superscripts $\sigma\tau$ from the notation, so that $U_{jk}^{\sigma\tau} = U_{jk}$. We observe that for j, k large enough (i.e., the support of $\chi(x, y)$ sufficiently small), we have on a dilate of the support $\mathscr{B}_{jk}$ of U_{jk}^{++}

$$|S''| \sim 2^{-(r+a_l s)\, j} 2^{-j \sum_{\mu=1}^{l-1} a_\mu N \left[{\cdot \atop \mu}\right]} \left| \Phi \begin{bmatrix} \cdot \\ l \end{bmatrix} \right| 2^{-a_l j \sum_{\mu=l+1}^n N \left[{\cdot \atop \mu}\right]}. \tag{4.16}$$

We need to introduce a finer partition in order to narrow down the range of values of $\Phi[{\cdot \atop l}]$. We select a root $r_\nu(x)$ in $\Phi[{\cdot \atop l}]$, say a root belonging to the grouping $\Phi[{\alpha \atop l}]$. It is not essential, but just convenient, to select the root with highest exponent $x^{a_{lL}^{\alpha}}$ among all of these.

If the root $r_\nu(x)$ is real (i.e., all coefficients in the Puiseux series expansion for $r_\nu(x)$ are real), we insert the partition of unity

$$1 = \sum_{m=-\infty}^{\infty} \sum_{\sigma = \pm} \chi(\sigma 2^m (y - r_\nu(x))). \tag{4.17}$$

in the integral formula for the operator U_{jk}. If the root $r_\nu(x)$ is complex, we insert rather

$$1 = \sum_{m=-\infty}^{\infty} \chi(2^m \, |y - r_\nu(x)|). \tag{4.18}$$

In either case, the operator U_{jk} has been decomposed further into a sum of operators $U_{jk;\,m}$, (we disregard the upper index σ in the case of real roots)

$$U_{jk} = \sum_{m \geqslant a_l j} \sum_{\sigma = \pm} U_{jk;\,m}^{\sigma}. \tag{4.19}$$

Evidently, the actual range of m in (4.18) is $m \geqslant a_l j$. We consider two separate ranges:

Contribution of the Range $m \gg a_l j$

First, we note that for $m \gg a_l j$, we have for $\beta \neq \alpha$

$$\left|\Phi\begin{bmatrix}\beta\\ l\end{bmatrix}\right| \sim 2^{-a_l jN\left[\begin{smallmatrix}\beta\\ l\end{smallmatrix}\right]}$$

so that the estimate (4.16) reduces to

$$|S''| \sim 2^{-j\sum_{\mu=1}^{l} a_\mu N\left[\begin{smallmatrix}\cdot\\ \mu\end{smallmatrix}\right]}\, 2^{a_l jN\left[\begin{smallmatrix}\alpha\\ l\end{smallmatrix}\right]} \left|\Phi\begin{bmatrix}\alpha\\ l\end{bmatrix}\right| 2^{-a_l j\sum_{\mu=l+1}^{n} N\left[\begin{smallmatrix}\cdot\\ \mu\end{smallmatrix}\right]}. \tag{4.20}$$

This means in effect that we are in exactly the same situation as when we had started, with however the full set of roots of $S''(x, y)$ replaced now by the smaller set of roots $\left[\begin{smallmatrix}\alpha\\ l\end{smallmatrix}\right]$

$$S''(x, y) \to \Phi\begin{bmatrix}\alpha\\ l\end{bmatrix}. \tag{4.21}$$

The role of the earlier exponents a_j is now taken over by the exponents $a^\alpha_{ll'}$, in the notation of (4.5). It is now easy to repeat the preceding construction, to narrow further the roots in the clusters. Thus we order the distinct exponents $a^\alpha_{ll'}$ in increasing order

$$a^\alpha_{l1} < a^\alpha_{l2} < \cdots < a^\alpha_{lL}. \tag{4.22}$$

By the same arguments as were used to study the contributions of the range $a_l j \ll k \ll a_{l+1} j$, we find that the contributions of the range $ja^\alpha_{lp} \ll m \ll ja^\alpha_{l(p+1)}$ are almost-orthogonal. Indeed, the arguments giving the size of $|S''_{xy}|$ on the support $\mathscr{B}_m$ of each operator $U_{jk;\,m}$ are identical, and $\mathscr{B}_m$ is obviously ribbon-like when the root $r_\nu(x)$ is real. When $r_\nu(x)$ is complex, write $|\mathrm{Im}\, r_\nu(x)| \sim |x|^b$, for some $b \geqslant a^\alpha_{lL}$. Then the condition $|y - r_\nu(x)| \sim 2^{-m}$ is equivalent to the condition $|y - \mathrm{Re}\, r_\nu(x)| \sim 2^{-m}$, so that the support of $U_{jk;\,m}$ is indeed ribbon-like, and Lemma 5 applies, giving almost-orthogonality. The same holds for the range $a^\alpha_{lL} \ll m \ll bj$, in complete analogy $a_n j \ll k$, without even the subtlety of having to join the two $\sigma = \pm$ regions, since this time we have summation with respect to a single index m, with respect to which the summands are almost-orthogonal.

Thus we need consider only the ranges $m \sim a^\alpha_{lp}$ for some p, and $m \sim bj$. We consider the first case. Recalling that the root $r_\nu(x)$ chosen in the decomposition (4.17–4.18) is actually the one of highest exponent a^α_{lL} among the a^α_{lp}'s, it is readily seen that all factors in $\Phi\left[\begin{smallmatrix}\alpha\\ l\end{smallmatrix}\right]$ are of definite size, except possibly for $\Phi\left[\begin{smallmatrix}\alpha & \cdot\\ l & p\end{smallmatrix}\right]$. This means we have carried out a reduction

$$\Phi\begin{bmatrix}\alpha\\ l\end{bmatrix} \to \Phi\begin{bmatrix}\alpha & \cdot\\ l & p\end{bmatrix}. \tag{4.23}$$

To complete a full reduction cycle, select a δ, and a root $r_\kappa(x)$ in $\Phi[\begin{smallmatrix}\alpha & \delta\\ l & p\end{smallmatrix}]$. We insert a new partition of unity

$$1=\sum_{\sigma=\pm}\sum_{M=-\infty}^{\infty}\chi(\sigma 2^M(y-r_\kappa(x)))$$

(again with suitable modifications for complex roots). This results in a decomposition $U_{jk;\,m}=\sum_M U_{jk;\,mM}$, where we have dropped the upper indices σ, since they do not play any longer any significant role. For $M\sim a^\alpha_{lp}$, this factor $(y-r_\kappa(x))$ is of definite size, and can be in effect peeled off, and we continue the reduction. For $M\gg a^\alpha_{lp}j$, it is the factors $\Phi[\begin{smallmatrix}\alpha & \gamma\\ l & p\end{smallmatrix}]$, for $\gamma\neq\delta$, which are of definite size $2^{-a^\alpha_{lp}j}$. Thus we have reduced

$$\Phi\begin{bmatrix}\alpha & \cdot\\ l & p\end{bmatrix}\to\Phi\begin{bmatrix}\alpha & \delta\\ l & p\end{bmatrix}. \tag{4.24}$$

This takes care of the case $m\sim a^\alpha_{lp}$. Turning now to the case $m\sim bj$, we may assume that $b>a^\alpha_{lL}$, otherwise this case falls within the previous one. We view the support of $U_{jk;\,m}$ as included in the ribbon-like box

$$\mathscr{B}_m=\{|y-\operatorname{Re} r_\nu(x)|\leqslant 2^{-bj},\ x\sim 2^{-j}\}.$$

It is easily seen that $|S''_{xy}|$ remains of constant size on a dilate of $\mathscr{B}_m$. Thus Lemma 5 establishes the almost-orthogonality of the summands. This completes our considerations for the case $m\gg a_l j$.

Contributions of the Range $m\sim a_l j$

By definition, the factor $|y-r_\nu(x)|$ is of definite size $2^{-a_l j}$. For j, k, large enough, this implies that the whole cluster $\Phi[\begin{smallmatrix}\alpha\\ l\end{smallmatrix}]$ is of definite size

$$2^{-a_l N[\begin{smallmatrix}\alpha\\ l\end{smallmatrix}]\,j}.$$

This reduces the cluster of (4.16) to

$$\Phi\begin{bmatrix}\cdot\\ l\end{bmatrix}\to\prod_{\beta\neq\alpha}\Phi\begin{bmatrix}\beta\\ l\end{bmatrix}.$$

We can iterate in this manner to arrive ultimately at a resolution of S''_{xy} where each factor is of definite size over a ribbon-like box, possibly after a finer decomposition of the type (3.14) (arising when all the scales m's of the successive decompositions coincide roughly with j multiplied by the exponents in the resolution). Our reduction argument is complete.

V. PROOF OF THEOREM 2

As mentioned in the Introduction, the proof of Theorem 2 is a routine adaptation of the proof of the case $\mu=0$ given in [5]. We shall therefore follow the outline of [5], and be brief. We assume that $\mu<\frac{1}{2}$. As in [5], the key steps occur in the treatment of some models, of which Model II, where

$$S''_{xy}=(y-x^{a_1})\cdots(y-x^{a_n}) \qquad 1\leqslant a_1<\cdots<a_n \tag{5.1}$$

and Model IV, where

$$S''_{xy}=(y-x^{a}-x^{b_1})\cdots(y-x^{a}-x^{b_N}) \qquad 1\leqslant a<b_1<\cdots<b_N \tag{5.2}$$

are of particular importance.

We discuss first the modifications needed in the treatment of Model II. As usual, we make a decomposition of D_μ into $D_{\mu;\,jk}$ with amplitude supported in $x\sim 2^{-j}$, $y\sim 2^{-k}$ (both x, y are assumed to be positive, for the sake of simplicity). Consider e.g. the resummation of the range $a_l j\ll k\ll a_{l+1}j$. Assume first that $B_l<A_l$, and set as in [5]

$$k=a_l j+r, \qquad A_l=a_1+\cdots+a_l, \qquad B_l=n-l.$$

We recall that the intersections of the (prolongations) of the faces of the Newton polyhedron are in this case given by $(\delta_l^{-1},\delta_l^{-1})$, with

$$\delta_l=\frac{1+a_l}{1+A_l+a_l(B_l+1)}.$$

The size and oscillating estimates for $D_{\mu;\,jk}$ are (c.f. (4.31) in [5])

$$\begin{aligned}\|D_{\mu;\,jk}\| &\leqslant 2^{-(1+a_l)j/2}2^{-r/2}(2^{-(A_l+a_lB_l)}2^{B_lr})^{\mu}\\ \|D_{\mu;\,jk}\| &\leqslant |\lambda|^{-1/2}\,2^{(1-2\mu)(A_l+a_lB_l)j/2}2^{(1-2\mu)\,B_lr/2}.\end{aligned} \tag{5.3}$$

The resummation method of [5] is based on estimating $\|D_{\mu;\,jk}\|$ by the convex combination θ which annihilates the j-factors,

$$(1-2\mu)(A_l+a_lB_l)\,\theta=(1-\theta)(1+a_l+2\mu(A_l+a_lB_l)).$$

This corresponds to the following value for θ

$$\theta = \frac{1 + a_l + 2\mu(A_l + a_l B_l)}{1 + A_l + (B_l + 1)\, a_l} = \delta_l + 2\mu(1 - \delta_l). \tag{5.4}$$

We need to check that $\theta \geqslant 0$. Now the function $x/1 - x$ is an increasing function of x for $0 < x < 1$. Since $\delta = \min_l \delta_l$, it follows that

$$\mu > -\frac{1}{2}\frac{\delta}{1-\delta} \geqslant -\frac{1}{2}\frac{\delta_l}{1-\delta_l} = -\frac{1}{2}\frac{1+a_l}{A_l + a_l B_l} \tag{5.5}$$

and θ is indeed non-negative. Substituting in (5.3), we obtain

$$\|D_{\mu;\, jk}\| \leqslant |\lambda|^{-(\delta_l + 2\mu(1-\delta_l))/2}\, 2^{-(1-\theta-B_l\theta+2B_l\mu)r/2}. \tag{5.6}$$

Since the terms $D_{\mu;\, jk}$ are almost-orthogonal for fixed r, the same bound (5.6) holds for the norm of the sum $\sum_{k = a_l j + r} D_{\mu;\, jk}$. Next, we need to check that the series in r is convergent. Now the exponent in r in (5.6) can be rewritten as

$$1 - \theta - B_l\theta + 2B_l\mu = (1 - \delta_l - B_l\delta_l)(1 - 2\mu). \tag{5.7}$$

Recall from [5] that the condition $B_l < A_l$ implies precisely that $1 - \delta_l - B_l\delta_l > 0$. Together with $\mu < \frac{1}{2}$, this establishes the strict positivity of (5.7), and hence the convergence of the r-series. We arrive in this way at the inequality

$$\sum_r \Big\| \sum_{k = a_l j + r} D_{\mu;\, jk} \Big\| \leqslant |\lambda|^{-(\delta_l + 2\mu(1-\delta_l))/2} \tag{5.8}$$

from which the desired bound $O(|\lambda|^{-(\delta + 2\mu(1-\delta))/2})$ follows, since

$$(\delta_l + 2\mu(1-\delta_l)) - (\delta + 2\mu(1-\delta)) = (1-2\mu)(\delta_l - \delta) \geqslant 0. \tag{5.9}$$

The same modifications for $\mu \neq 0$ suffice to treat the cases $A_l < B_l$ and $A_l = B_l$, as well as the remaining summations in Model II.

We discuss next the modifications required for Model IV. In this case, the explicit value of δ is

$$\delta = \frac{1+a}{1+a+Na}.$$

After the partition of D_μ into $D_{\mu;\, jk}$, it is easy to see that the only difficulties reside in the range $k \sim aj$, for which it suffices, by orthogonality, to establish the uniform boundedness of $\|D_{\mu;\, jk}\|$. As in [5], we need to decompose $D_{\mu;\, jk}$ further into $D^m_{\mu;\, jk}$, with $y - x^a - x^{b_N} \sim 2^{-m}$. The resummation in m involves in particular the range $m \gg b_{N-1} j$ and $b_p j \ll m \ll b_{p+1} j$, for which the necessary modifications are as follows.

Let $m = b_{N-1} j + r$. The bounds for $D^m_{\mu;\, jk}$ are

$$\begin{aligned} \|D^m_{\mu;\, jk}\| &\leqslant 2^{-(1+\mu) r} 2^{-[1-a+2\mu(b_1+\cdots+b_{N-1})+2(1+\mu) b_{N-1}]\, j/2} \\ \|D^m_{\mu;\, jk}\| &\leqslant |\lambda|^{-1/2}\, 2^{(1-2\mu)((b_1+\cdots+b_{N-1})+b_{N-1})\, j/2} 2^{(1-2\mu) r/2}. \end{aligned} \tag{5.10}$$

The convex combination θ of the two estimates in (5.10) annihilating the j-factors is

$$\theta = \frac{1+2(1+\mu)\, B_{N-1} - a + 2\mu(b_1 + \cdots + b_{N-1})}{1 + b_1 + \cdots + b_{N-1} + 3b_{N-1} - a}. \tag{5.11}$$

The arguments of Lemma 4 in [5] can be adapted to show that

$$\theta \geqslant \delta + 2\mu(1-\delta). \tag{5.12}$$

More specifically, if we introduce for each p the following function of b

$$\theta_p(b) = \frac{1 + 2(1 + (N+1-p)\mu) b - a + 2\mu(b_1 + \cdots + b_{p-1}))}{1 + b_1 + \cdots + b_{p-1} + (N+3-p) b - a}, \tag{5.13}$$

then, up to a positive factor, $\mathrm{d}\theta/\mathrm{d}b = (1-2\mu)((N+1-p)(a-1) + 2(b_1 + \cdots + b_{p-1})$. In particular, the function $\theta_p(b)$ is increasing, and we have

$$\begin{aligned} \theta &= \theta_{N-1}(b_{N-1}) \geqslant \theta_{N-1}(b_{N-2}) = \theta_{N-2}(b_{N-2}) \geqslant \cdots \geqslant \theta_1(b_1) \\ &= \frac{1 + 2(1+N\mu) b_1 - a}{1 + (N+2) b_1 - a}. \end{aligned} \tag{5.14}$$

For $a > 1$, the function $\theta_1(x)$ is a strictly increasing function of x, and we have $\theta_1(b_1) > \theta_1(a) = \delta + 2\mu(1-\delta)$. For $a = 1$, we have identically $\theta_1(b) = \delta + 2\mu(1-\delta)$ for all b. This establishes (5.12), and in particular, that $\theta \geqslant 0$.

Again, we have to verify that the resulting series in r is geometrically convergent. This is the case, provided $\theta < \frac{2}{3}(1+\mu)$, which is equivalent to

$$(1-2\mu)(a - 1 + 2(b_1 + \cdots + b_{N-1})) > 0. \tag{5.15}$$

This is true when $\mu < \frac{1}{2}$ and $a > 1$ or $a = 1$ and $b_1 + \cdots + b_{N-1} > 0$ (c.f. Lemma 3 of [5]).

We consider now the range $b_p j \ll m \ll b_{p+1} j$, and set $m = [b_p j] + r$, $r = jM$. The estimates for $D^m_{\mu;\, jk}$ are (c.f. (4.58) of [5])

$$\begin{aligned} \|D^m_{\mu;\, jk}\| &\leqslant 2^{-[1-a+2\mu(b_1+\cdots+b_p)+(1+\mu(N-p))(b_p+M)]\, j/2} \\ \|D^m_{\mu;\, jk}\| &\leqslant |\lambda|^{-1/2}\, 2^{(1-2\mu)(b_1+\cdots+b_p+(b_p+M)(N-p))\, j/2}. \end{aligned} \tag{5.16}$$

Define θ_M^* to be the convex combination annihilating the j-factors. It is given by the following expression, akin to (5.11)

$$\theta_M^* = \frac{1+2(1+\mu(N-p))(b_p+M)-a+2\mu(b_1+\cdots+b_p)}{1+b_1+\cdots+b_p+(2+N-p)(b_p+M)-a}. \tag{5.17}$$

Unless $p=1$ and $a=1$ (which case is easy and can be treated explicitly, as in (4.61) in [5]), we have $\theta_M^* > \delta+2\mu(1-\delta)$. As in [5], we choose rather a convex combination θ_M of the two estimates in (5.16) with

$$\theta_M = \theta_M^* - \varepsilon > \delta + 2\mu(1-\delta)$$

for a small positive ε. In terms of r, the resulting estimate for $D^m_{\mu;\, jk}$ is

$$\begin{aligned}\|D^m_{\mu;\, jk}\| \leqslant |\lambda|^{-\theta_M/2}\, & 2^{-(1+b_1+\cdots+b_p+(1+\mu(N-p)+(1-2\mu)(N-p))b_p-a)\,\varepsilon j/2} \\ & \times 2^{-(1+\mu(N-p)+(1-2\mu)(N-p))\,\varepsilon r/2}. \end{aligned} \tag{5.18}$$

We note that $\mu(N-p)+(1-2\mu)(N-p)=(1-\mu)(N-p)$ is manifestly strictly positive. Thus the bounds in j are in (5.18) are all less than 1, and the series in r is geometrically convergent, giving the even better estimate $O(|\lambda|^{-\theta_M/2})$.

With the preceding modifications for the Model cases II and IV when $\mu \neq 0$, the formalism introduced in [5] applies then verbatim, giving Theorem 2.

VI. PROOF OF THEOREM 3

We require the following lemmas on the L^2 boundedness of operators with positive kernels $K(x, y)$.

LEMMA 6. *Assume that there exists a constant C so that*

$$\int_I K(x,y)\, y^{-1/2}\, \mathrm{d}y \leqslant Cx^{-1/2}, \qquad \int_I K(x,y)\, x^{-1/2}\, \mathrm{d}x \leqslant Cy^{-1/2},$$

where I is any interval in $\mathbf{R}_+$. Then the operator $f \to \int_I K(x,y)\, f(y)\, \mathrm{d}y$ is bounded on $L^2(I)$.

Indeed, it suffices to show that the expression

$$\int_I \int_I K(x,y)\, |f(y)| \cdot |g(x)|\, \mathrm{d}y\, \mathrm{d}x \tag{6.1}$$

is uniformly bounded over all functions f, g with $\|f\|_{L^2(I)} = \|g\|_{L^2(I)} = 1$. This is easy to see if we estimate $|f(y)| \cdot |g(x)|$ in (6.1) by

$$|f(y)| \cdot |g(x)| \leqslant \tfrac{1}{2}(|f(y)|^2\, x^{-1/2} y^{1/2} + |g(x)|^2\, y^{-1/2} x^{1/2})$$

and apply Fubini's theorem to the resulting two terms.

LEMMA 7. *The operators*

(i) $x^{-N} \int_0^{x^b} y^{-M} f(y)\, dy$,

(ii) $x^{-N} \int_{x^a}^{\infty} y^{-M} f(y)\, dy$,

(iii) $x^{-N} \int_{x^a}^{x^b} y^{-M} f(y)\, dy$,

(iv) $x^{-N} \int_{cx^a}^{Cx^a} |y - x^a|^{-M}\, f(y)\, dy$, *with* $c < C$,

are bounded on $L^2(\mathbf{R}_+)$ *respectively when*

(i) $(N - \frac{1}{2}) + (M - \frac{1}{2}) b = 0$, *and* $M < \frac{1}{2}$;

(ii) $(N - \frac{1}{2}) + (M - \frac{1}{2}) a = 0$, *and* $M > \frac{1}{2}$;

(iii) $(N - \frac{1}{2}) + (M - \frac{1}{2}) b = 0$, *and* $M < \frac{1}{2}$, *or* $(N - \frac{1}{2}) + (M - \frac{1}{2}) a = 0$, *and* $M > \frac{1}{2}$;

(iv) $(N - \frac{1}{2}) + (M - \frac{1}{2}) a = 0$, *and* $M < 1$.

The less stringent conditions

(i)′ $(N - \frac{1}{2}) + (M - \frac{1}{2}) b \leqslant 0$, *and* $M < \frac{1}{2}$,

(ii)′ $(N - \frac{1}{2}) + (M - \frac{1}{2}) a \leqslant 0$, *and* $M > \frac{1}{2}$,

(iii)′ $(N - \frac{1}{2}) + (M - \frac{1}{2}) b \leqslant 0$, *and* $M < \frac{1}{2}$, *or* $(N - \frac{1}{2}) + (M - \frac{1}{2}) a \leqslant 0$, *and* $M > \frac{1}{2}$,

(iv)′ $(N - \frac{1}{2}) + (M - \frac{1}{2}) a \leqslant 0$, *and* $M < 1$,

suffice to insure their respective boundedness on $L^2(I)$, *for any bounded interval* $I \subset \mathbf{R}_+$.

To prove Lemma 7, it suffices in view of Lemma 6 to show that the operators and their adjoints map the function $f(y) = y^{-1/2}$ to a function bounded by a multiple of f. (Observe that the adjoint of (i) is (ii), with $a = b^{-1}$, and N, M unchanged). This is easily done under the given hypotheses.

LEMMA 8. *Let* $r_1(x), ..., r_n(x)$ *be* n *functions satisfying* $cx^a < r_l(x) < Cx^a$, $1 \leqslant l \leqslant n$. *Then the operator*

$$f \to \int_{cx^a}^{Cx^a} \left| \prod_{l=1}^{n} (y - r_l(x)) \right|^{-M/n} f(y)\, dy \tag{6.2}$$

is bounded on $L^2(I)$ *for* $(N-\frac{1}{2})+(M-\frac{1}{2})a \leqslant 0$ *and* $M<1$. *Here* I *is a bounded interval in* $\mathbf{R}_+$.

The case $n=1$ is an easy modification of (iv) in Lemma 7. The general case follows from the Hölder inequality

$$\int \left| \prod_{l=1}^{n} F_l(y) \right| dy \leqslant \prod_{l=1}^{n} \left(\int |F_l(y)|^n \, dy \right)^{1/n}$$

with $F_l(y) = |y - r_l(x)|^{-M/n} \, |f(y)|^{1/n}$.

We turn now to the proof of Theorem 3. We adopt for $\Psi(x, y)$ the same factorization, Newton diagram, notation, as used previously for $S''_{xy}(x, y)$, e.g.,

$$\Psi(x, y) = U(x, y)\, x^r y^s \prod_{\nu} (y - r_\nu(x)) = \prod_{l=1}^{n} \Phi \begin{bmatrix} \cdot \\ l \end{bmatrix}, \tag{6.3}$$

with $\Phi[{}^{\cdot}_{l}]$ the product of all factors corresponding to roots with leading exponent $c_l^\alpha x^{a_l}$ in their Puiseux series expansion, etc. In particular, recall (c.f. (4.6)) that $N[{}^{\cdot}_{l}]$ denotes the number of factors in $\Phi[{}^{\cdot}_{l}]$ (generalized multiplicity of the exponent a_l). As in [5], it is convenient to introduce the quantities

$$A_l = r + \sum_{j=1}^{l} a_j N \begin{bmatrix} \cdot \\ j \end{bmatrix}, \qquad B_l = s + \sum_{j=l+1}^{n} N \begin{bmatrix} \cdot \\ j \end{bmatrix}. \tag{6.4}$$

We recall (see, e.g., [5]) that the vertices of the Newton diagram of $\Psi(x, y)$ are (A_l, B_l), and that the intersections of the faces of the Newton diagram with the bisectrix $p=q$ are $(\delta_l^{-1}, \delta_l^{-1})$ with

$$\delta_l = \frac{1+a_l}{A_l + a_l B_l}. \tag{6.5}$$

(These expressions are not to be confused with the ones preceding (5.3), where δ_l referred to the reduced Newton diagram for $S(x, y)$, while A_l, B_l referred rather to the roots of S''_{xy}. In the present context, both δ_l and A_l, B_l refer to the Newton diagram and the roots of the same function $\Psi(x, y)$.)

As in the proof of Theorems 1 and 2, we can assume for simplicity of notation that all roots $r_\nu(x)$ are real, with the general case an easy adaptation of this case, using the arguments of Section IV.(f) of [5]. By considering separately the four quadrants in the (x, y) plane, we also note that it suffices to consider the case of x, y small and positive, and thus to establish the boundedness of the operator (1.7) as an operator on $L^2(I)$, where I is a small interval near the origin in $\mathbf{R}_+$.

By restricting I to be small enough, we may assume that there exists constants $c_l < C_l$ so that

$$c_l x^{a_l} < r_\nu(x) < C_l x^{a_l}; \qquad C_{l+1} x^{a_{l+1}} < c_l x^{a_l}, \tag{6.6}$$

for $x \in I$, and all roots $r_\nu(x)$ in $\Phi[{}^{\cdot}_{l}]$ (with leading coefficients $c_l^\alpha x^{a_l}$ with $c_l^\alpha > 0$; when c_l^α is negative or complex, the factor $|y - r_\nu(x)|^{-M}$ will satisfy even better bounds than the ones we need below). Thus we assume *all* roots $r_\nu(x)$ satisfy (6.6).

In this way, we can divide $I \times I$ into regions of the form $c_l x^{a_l} < y < C_l x^{a_l}$, $c_{l+1} x^{a_l+1} < y < c_l x^{a_l}$, $0 < y < c_n x^{a_n}$, and $C_1 x^{a_1} < y < |I|$, with resulting operators

(1) $\int_{c_l x^{a_l}}^{C_l x^{a_l}} |\Psi(x, y)|^{-\mu} f(y)\, dy$;

(2) $\int_{c_{l+1} x^{a_l+1}}^{c_l x^{a_l}} |\Psi(x, y)|^{-\mu} f(y)\, dy$;

(3) $\int_0^{c_n x^{a_n}} |\Psi(x, y)|^{-\mu} f(y)\, dy$;

(4) $\int_{C_1 x^{a_1}}^{|I|} |\Psi(x, y)|^{-\mu} f(y)\, dy$.

We observe that the above divisions do not require smooth partitions of unity. In fact, all our arguments are based on the size of $|\Psi(x, y)|$ only, and involve for example no integration by parts. Finally, we may assume further that $a_l \geqslant 1$ for all l. Indeed, the regions with $a_l < 1$ can be treated in the same way, but by factoring $\Psi(x, y)$ as a product of $x - \tilde{r}_\rho(y)$, so that the leading exponents in the Puiseux expansions for $\tilde{r}_\rho(y)$ are greater than 1.

We consider first the operator in (1). In this range, we have

$$|\Psi(x, y)| \sim x^{-(A_{l-1} + a_l B_l)\Phi\left[\begin{smallmatrix}\cdot \\ l\end{smallmatrix}\right]}$$

and thus we have

$$\left| \int_{c_l x^{a_l}}^{C_l x^{a_l}} |\Psi(x, y)|^{-\mu} f(y)\, dy \right|$$

$$\leqslant C x^{-\mu(A_{l-1} + a_l B_l)} \int_{c_l x^{a_l}}^{C_l x^{a_l}} \prod_{\nu \in \Phi[{}^{\cdot}_{l}]} |y - r_\nu(x)|^{-\mu} |f(y)|\, dy.$$

In view of Lemma 8, this operator is bounded on $L^2(I)$ when $\mu N[{}^{\cdot}_{l}] < 1$, and

$$\mu(A_{l-1} + a_l B_l) - \frac{1}{2} + \left[\mu N\left[\begin{matrix}\cdot \\ l\end{matrix}\right] - \frac{1}{2}\right] a_l \leqslant 0. \tag{6.7}$$

The condition (6.7) is easily seen to be equivalent to $\mu \leqslant \frac{1}{2}\delta_l$, which is satisfied, since $\mu \leqslant \frac{1}{2}\delta_0 = \frac{1}{2}\min_l \delta_l$. Now (6.7) implies $\mu N_l < 1$, unless $\frac{1}{2}\delta_l \geqslant N_l^{-1}$. However, this last equation can be rewritten as

$$2A_{l-1} + (a_l - 1)\, N_l + 2a_l B_l \leqslant 0. \tag{6.8}$$

Since $N_l \geqslant 1$, this cannot happen when $a_l > 1$. When $a_l = 1$, the only case where it can happen is when we have in addition $A_{l-1} = B_l = 0$, i.e., there is only one generalized exponent $a_l = a_1$, and no factor $x^r y^s$. This is the case where the main face of the Newton diagram is of equation $p + q = constant$, which we ruled out in condition (C) of Theorem 3.

Next, in the range $C_{l+1} x^{a_{l+1}} < y < c_l x^{a_l}$, we have

$$|\Psi(x, y)| \sim x^{A_l} y^{B_l}.$$

This means that we can apply Lemma 7, and find that we have boundedness under either one of the following two sets of conditions

$$\begin{aligned} &\mu \leqslant \tfrac{1}{2}\delta_l, && B_l \mu < \tfrac{1}{2}; \\ &\mu \leqslant \tfrac{1}{2}\delta_{l+1}, && B_l \mu > \tfrac{1}{2}. \end{aligned} \tag{6.9}$$

Since $\delta_0 \leqslant \min_l \delta_l$, it follows that we have boundedness as long as $\mu B_l \neq \frac{1}{2}$ for some l. If $\mu < \frac{1}{2}\delta_0$, we can clearly replace μ by a slightly larger $\tilde{\mu}$ which still satisfies $\tilde{\mu} < \frac{1}{2}\delta_0$, but for which $\tilde{\mu} B_l \neq \frac{1}{2}$ for any l. Assume then that $\mu = \frac{1}{2}\delta_0$, and that $\mu B_l = \frac{1}{2}$ for all l. This implies that the point (B_l, B_l) must be the intersection of the bisectrix $p = q$ with the main face of the Newton diagram. On the other hand, the inequality $\mu \leqslant \frac{1}{2}\delta_l$ implies that $A_l \leqslant B_l$. If $A_l = B_l$, this means that the main face is a point vertex, which is ruled out by the hypotheses of Theorem 3. Otherwise, $A_l < B_l$, and the half-line $\{(p, q);\, p \geqslant A_l,\, q = B_l\}$ must lie within the Newton polyhedron. But it must be on its boundary, for otherwise the boundary will intersect the line $p = q$ at a point nearer to the origin than (B_l, B_l). Thus the line $p = q$ intersects the bisectrix $p = q$ at a horizontal main face, a situation which was also ruled out by the hypotheses of Theorem 3.

We consider now the range (3), namely $0 < y < c_n x^{a_n}$. We have then

$$|\Psi(x, y)| \sim x^{A_n} y^{B_n}$$

(note that $B_n = s$). We can apply (iii) from Lemma 7, which shows that we have boundedness when $\mu \leqslant \frac{1}{2}\delta_n$ and $B_n \mu < \frac{1}{2}$. The first condition is of course satisfied for $\mu = \frac{1}{2}\min \delta_l$. The second will be a consequence of the first, unless $\delta_n \geqslant B_n^{-1}$. This works out to be

$$B_n \geqslant A_n. \tag{6.10}$$

However, (A_n, B_n) is the point of lowest q coordinate among the vertices of the Newton diagram, and the half-line $\{(p, q);\, p \geqslant A_n, q = B_n\}$ is the horizontal face. The condition (6.10) says that the vertex (A_n, B_n) must be on the left side of the bisectrix $p = q$, which must then intersect the horizontal face at the point (B_n, B_n). This situation was ruled out in the hypotheses of Theorem 3.

Finally, we can easily verify that the range (4) can be treated with the help of (iv) of Lemma 7, in complete analogy with (3), with the hypothesis that the main face not be a vertex or a vertical face insuring the requirements listed in Lemma 7. The proof of Theorem 3 is complete.

Remark. After this work was completed, we received the preprint [7], which deals with some related results for the C^∞ case, with loss of ε.

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